Passivity-Based Gain-Scheduled Control with Scheduling Matrices

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August 21, 2024

Motivating Example

Consider the control of the rigid two-link robotic manipulator in Figure 1.

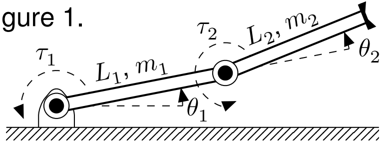


Figure 1: Two-link robotic manipulator.

For simplicity, the robot is assumed to be:

1. Planar.
2. Fixed at the base.

The equations of motion of this two-link robot are given by

$$\mathbf{M}(\mathbf{q}(t))\ddot{\mathbf{q}}(t) = \mathbf{f}_{\text{non}}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) + \mathbf{u}(t),$$

where

- $\mathbf{M}(\mathbf{q}(t)) = \mathbf{M}^\mathsf{T}(\mathbf{q}(t)) \succ 0$ is the mass matrix,
- $\mathbf{f}_{\text{non}}(\mathbf{q}(t), \dot{\mathbf{q}}(t))$ captures the nonlinear inertial and Coriolis forces,
- $\mathbf{u}(t) = \begin{bmatrix}\tau_1(t) & \tau_2(t)\end{bmatrix}^\mathsf{T}$ are the joint torques, and
- $\mathbf{q}(t) = \begin{bmatrix}\theta_1(t) & \theta_2(t)\end{bmatrix}^\mathsf{T}$ are the generalized coordinates.

Control Objective

The control objective is to have the two-link robot track the position and rate trajectory in Figure 2.

Nonlinear System

$$\mathbf{M}(\mathbf{q}(t))\ddot{\mathbf{q}}(t) = \mathbf{f}_{\text{non}}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) + \mathbf{u}(t).$$

The mapping between the joint torques to joint rates, $\mathbf{u}(t) \to \dot{\mathbf{q}}(t)$, is **passive**.

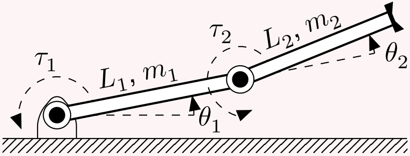


Figure 1: Rigid two-link robotic manipulator.

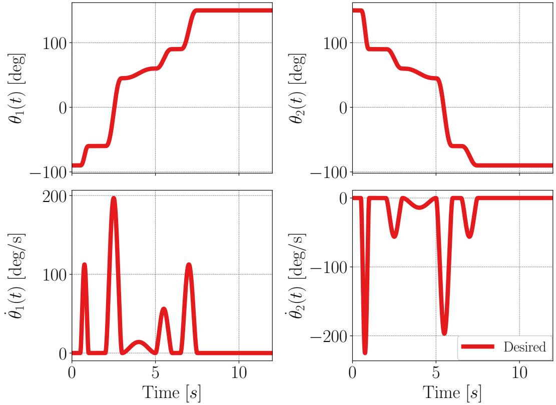


Figure 2: Desired angle position and rate.

What is Passivity?

Definition (Passivity [Márquez, 2003])

Consider a square system with input $\mathbf{u} \in \mathcal{L}_{2e}$ and output $\mathbf{y} \in \mathcal{L}_{2e}$ mapped through the operator $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$. The system $\mathcal{G}$ is

- **passive** if $\exists \beta \in \mathbb{R}_{\leq 0}$ s.t.

$$\langle \mathbf{y}, \mathbf{u} \rangle_T \geq \beta, \qquad \forall \mathbf{u} \in \mathcal{L}_{2e}, \quad \forall T \in \mathbb{R}_{\geq 0},$$

- **very strictly passive (VSP)** if $\exists \delta, \varepsilon \in \mathbb{R}_{>0}$ and $\exists \beta \in \mathbb{R}_{\leq 0}$ s.t.

$$\langle \mathbf{y}, \mathbf{u} \rangle_T \geq \beta + \delta \left\| \mathbf{u} \right\|_{2T}^2 + \varepsilon \left\| \mathbf{y} \right\|_{2T}^2, \qquad \forall \mathbf{u} \in \mathcal{L}_{2e}, \quad \forall T \in \mathbb{R}_{\geq 0},$$

- **input strictly passive (ISP)** if $\delta \in \mathbb{R}_{>0}$ and $\varepsilon = 0$,
- **output strictly passive (OSP)** if $\varepsilon \in \mathbb{R}_{>0}$ and $\delta = 0$.

Why Should You Care?

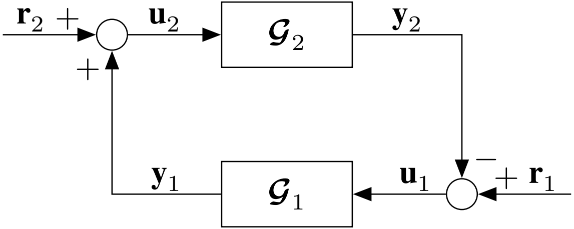


Figure 3: The negative feedback interconnection of two systems $\mathcal{G}_1$ and $\mathcal{G}_2$.

Theorem (Passivity Theorem [Márquez, 2003])

*Consider the negative feedback interconnection of $\mathcal{G}_1 : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ and $\mathcal{G}_2 : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ in Figure 3. Provided $\mathcal{G}_1$ is **passive** and $\mathcal{G}_2$ is **VSP**, the negative feedback interconnection is $\mathcal{L}_2$-stable.*

VSP Controller Synthesis

- **Strictly positive real (SPR)** controllers with **feedthrough** are in turn **VSP** [Márquez, 2003].
- The **Kalman-Yakubovich-Popov (KYP) lemma** and gain matrix from the **linear-quadratic regulator (LQR)** problem can be used to synthesize the SPR controllers [Benhabib et al., 1981].
- LQR problem needs the linearized model of the system dynamics about a **linearization point.**

Question

How to choose the linearization point?

Question

How good is the linearized model away from this linearization point?

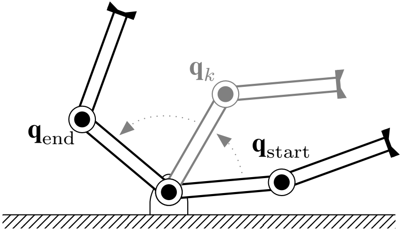
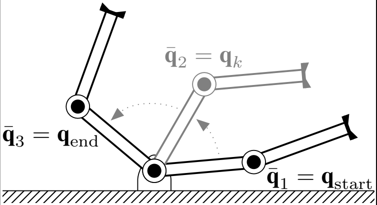


Figure 4: Two-link robotic manipulator trajectory.

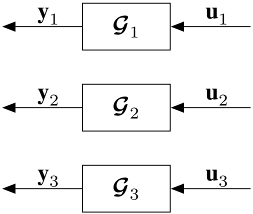
Gain-Scheduling Linear VSP Controllers

(a) Linearize the system about $N = 3$ points



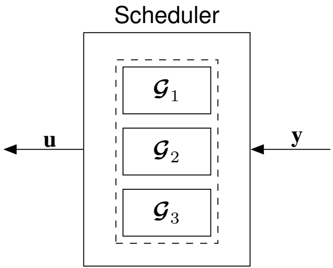
$$\delta\dot{\mathbf{x}}(t) = \mathbf{A}_i \delta\mathbf{x}(t) + \mathbf{B}_i \delta\mathbf{u}(t)$$

(b) Synthesize linear controllers



$$\dot{\mathbf{x}}_i(t) = \mathbf{A}_{c,i}\mathbf{x}_i(t) + \mathbf{B}_{c,i}\mathbf{u}_i(t)$$
$$\mathbf{y}_i(t) = \mathbf{C}_{c,i}\mathbf{x}_i(t) + \mathbf{D}_{c,i}\mathbf{u}_i(t)$$

(c) Gain-schedule the controllers



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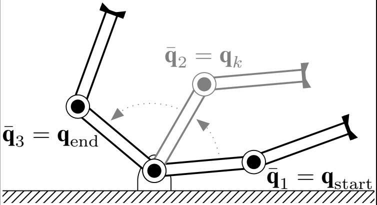

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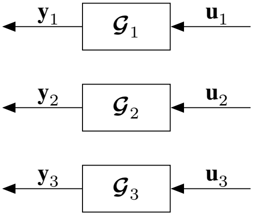

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}_{c,i}\mathbf{x}_i(t) + \mathbf{B}_{c,i}\mathbf{u}_i(t)$$
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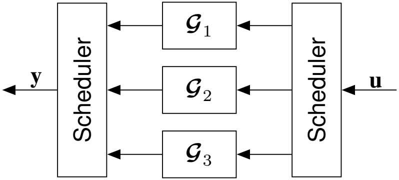


Question

What are the passivity properties of the gain-scheduled system?

Passivity Preserving Gain-Scheduling Architecture

Within the context of **passivity**-based control, the gain-scheduling architecture in Figure 5 was proposed in [Damaren, 1996].

Scheduling is achieved via the scalar scheduling function $s_i(\boldsymbol{\zeta}(t), \mathbf{x}(t), t)$, where

- $\mathbf{x}(t)$ is the state of the plant under control,
- $\boldsymbol{\zeta}(t)$ represent any external signal convenient for scheduling.

The gain-scheduled controller is

- **passive** if the subcontrollers are passive,
- **ISP** if the subcontrollers are ISP.

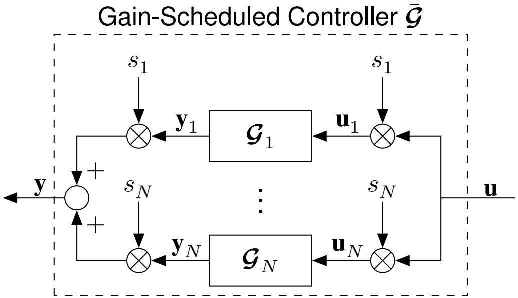
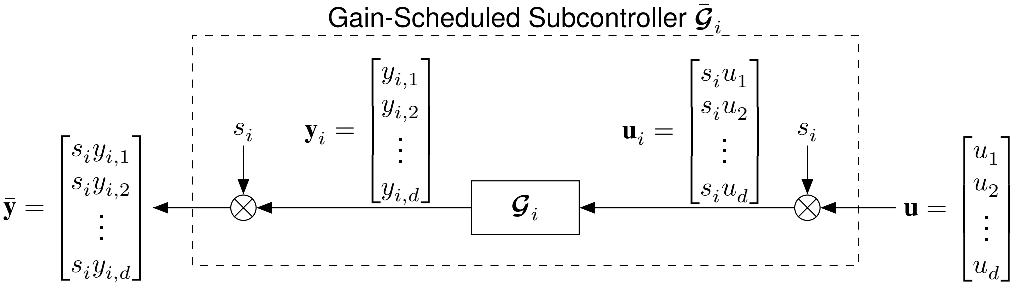


Figure 5: Gain-scheduling architecture [Damaren, 1996]. The node $\otimes$ performs scalar-vector multiplication using the scalar scheduling signal $s_i(\boldsymbol{\zeta}(t), \mathbf{x}(t), t)$.

Passivity Preserving Gain-Scheduling Literature

- *Gain-Scheduled SPR Controllers for Nonlinear Flexible Systems* [Damaren, 1996]
- *Design of Gain-Scheduled Strictly Positive Real Controllers Using Numerical Optimization for Flexible Robotic Systems* [Forbes, Damaren, 2010]
- *A Very Strictly Passive Gain-Scheduled Controller: Theory and Experiments* [Walsh, Forbes, 2016]
- *Very Strictly Passive Controller Synthesis With Affine Parameter Dependence* [Walsh, Forbes, 2018]
- *Gain-Scheduled Control for an Antenna with Multiple Collocated Sensors and Actuators* [Lang, Damaren, 2018]

Scalar-Gain-Scheduling Architecture

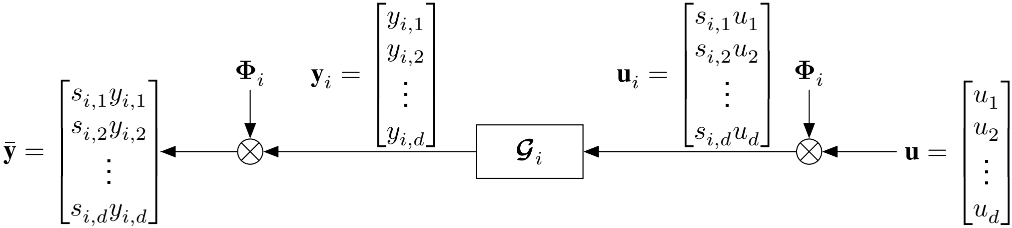


Question

- Why should $(u_1, y_{i,1}), (u_2, y_{i,2}), \ldots, (u_d, y_{i,d})$ all be scheduled by the **same** s_i?

Gain-Scheduling Goal

Consider the case where the scheduling function is a **diagonal matrix**, $\mathbf{\Phi}_i = \mathrm{diag}(s_{i,1}, \cdots, s_{i,d})$:



$$\bar{\mathbf{y}} = \begin{bmatrix} s_{i,1}y_{i,1} \\ s_{i,2}y_{i,2} \\ \vdots \\ s_{i,d}y_{i,d} \end{bmatrix}, \quad \mathbf{y}_i = \begin{bmatrix} y_{i,1} \\ y_{i,2} \\ \vdots \\ y_{i,d} \end{bmatrix}, \quad \mathbf{u}_i = \begin{bmatrix} s_{i,1}u_1 \\ s_{i,2}u_2 \\ \vdots \\ s_{i,d}u_d \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{bmatrix}$$

Questions

- Can the scheduling function be **any matrix** (not necessarily diagonal)?
- Can the input scheduling function be **different** from the output scheduling function?

Novel Matrix-Gain-Scheduling Architecture

- Consider subcontrollers $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N$ of the form $\mathbf{y}_i(t) = (\mathcal{G}_i \mathbf{u}_i)(t)$, for $i \in \mathcal{N} = \{1, \ldots, N\}$.
- The subcontrollers could be **linear** or **nonlinear**.
- The input-output map of $\bar{\mathcal{G}}$ is given by

$$\mathbf{u}_i(t) = \mathbf{\Phi}_i(\boldsymbol{\zeta}(t), \mathbf{x}(t), t)\mathbf{u}_c(t), \tag{1a}$$

$$\mathbf{y}_c(t) = \sum_{i \in \mathcal{N}} \alpha_i \mathbf{\Phi}_i^{\mathsf{T}}(\boldsymbol{\zeta}(t), \mathbf{x}(t), t)\mathbf{y}_i(t), \tag{1b}$$

for $\mathbf{u}_c, \mathbf{y}_c \in \mathbb{R}^n$, $\alpha_i \in \mathbb{R}_{>0}$, and $\mathbf{\Phi}_i \in \mathbb{R}^{n \times n}$ for all $i \in \mathcal{N}$.

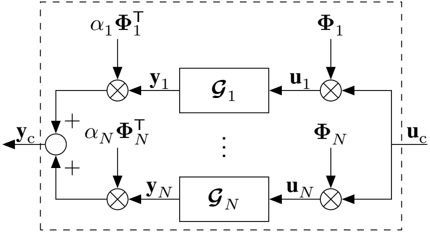


Figure 6: Matrix-gain-scheduling architecture. The node $\otimes$ performs matrix-vector multiplication using the scheduling matrices $\mathbf{\Phi}_i(\boldsymbol{\zeta}(t), \mathbf{x}(t), t)$. The positive constants α_i are used to scale the gain.

Question

Knowing the subcontrollers, $\mathcal{G}_i$, have certain passivity properties, what can be said about the **passivity properties of the matrix-gain-scheduled controller**, $\bar{\mathcal{G}}$?

Scheduling Matrix Properties

Definition (Active Scheduling Matrices)

The scheduling matrices are said to be:

- **Active** if at all times, there exists **at least one nonzero scheduling matrix**.
- **Strongly active** if at all times, there exists **at least one full rank scheduling matrix**.

Lemma

Provided the scheduling matrices are strongly active, then

$$\sum_{i\in\mathcal{N}} \lambda_{\min}\left(\mathbf{\Phi}_i^{\mathsf{T}}(t)\mathbf{\Phi}_i(t)\right) = \sum_{i\in\mathcal{N}} \nu_i^2(t) > 0, \quad \forall t \in [0, T],$$

where $\nu_i(t)$ *is the smallest singular value of* $\mathbf{\Phi}_i(t)$*.*

Why have $\mathbf{\Phi}_i$ and $\mathbf{\Phi}_i^{\mathsf{T}}$?

Review (Input-Output Map of $\bar{\mathcal{G}}$)

The input-output map of $\bar{\mathcal{G}}$ is given by

$$\mathbf{u}_i(t) = \mathbf{\Phi}_i(t)\mathbf{u}_{\mathrm{c}}(t), \tag{1a}$$

$$\mathbf{y}_{\mathrm{c}}(t) = \sum_{i \in \mathcal{N}} \alpha_i \mathbf{\Phi}_i^{\mathsf{T}}(t)\mathbf{y}_i(t). \tag{1b}$$

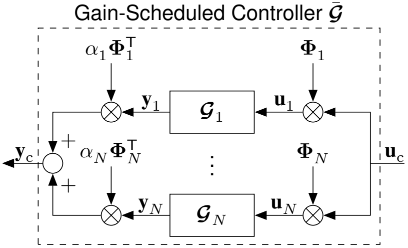


Figure 6: Matrix-gain-scheduling architecture.

Remark

Using the input-output map of the gain-scheduled controller $\bar{\mathcal{G}}$ in (1), it follows that

$$\langle \mathbf{u}_{\mathrm{c}}, \mathbf{y}_{\mathrm{c}} \rangle_T = \sum_{i \in \mathcal{N}} \int_0^T \alpha_i \mathbf{u}_{\mathrm{c}}^{\mathsf{T}}(t)\mathbf{\Phi}_i^{\mathsf{T}}(t)\mathbf{y}_i(t)\,\mathrm{d}t = \sum_{i \in \mathcal{N}} \int_0^T \alpha_i (\mathbf{\Phi}_i(t)\mathbf{u}_{\mathrm{c}}(t))^{\mathsf{T}}\mathbf{y}_i(t)\,\mathrm{d}t = \sum_{i \in \mathcal{N}} \alpha_i \langle \mathbf{u}_i, \mathbf{y}_i \rangle_T .$$

Main Contribution

Theorem (Gain-Scheduling of ISP subcontrollers)

Consider N *ISP subcontrollers* $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N$ *of the form* $\mathbf{y}_i(t) = (\mathcal{G}_i \mathbf{u}_i)(t)$ *satisfying*

$$\langle \mathbf{u}_i, \mathbf{y}_i \rangle_T \geq \beta_i + \delta_i \|\mathbf{u}_i\|_{2T}^2, \qquad \forall \mathbf{u}_i \in \mathcal{L}_{2e}, \quad \forall T \in \mathbb{R}_{\geq 0},$$

for $i \in \mathcal{N} = \{1, \ldots, N\}$ *with* $\beta_i \in \mathbb{R}_{\leq 0}$ *and* $\delta_i \in \mathbb{R}_{>0}$.

Provided the scheduling matrices are ***strongly active***, *the gain-scheduled controller* $\bar{\mathcal{G}}$ *is ISP with*

$$\langle \mathbf{u}_c, \mathbf{y}_c \rangle_T \geq \hat{\beta} + \hat{\delta} \|\mathbf{u}_c\|_{2T}^2, \qquad \forall \mathbf{u}_c \in \mathcal{L}_{2e}, \quad \forall T \in \mathbb{R}_{\geq 0},$$

where

$$\hat{\beta} = \sum_{i \in \mathcal{N}} \alpha_i \beta_i \leq 0, \qquad \hat{\delta} = \delta_{\min} \nu_{\inf} > 0, \qquad \delta_{\min} = \min_{i \in \mathcal{N}} \alpha_i \delta_i > 0, \qquad \nu_{\inf} = \inf_{t \in [0,T]} \sum_{i \in \mathcal{N}} \nu_i^2(t) > 0.$$

Main Contribution

Theorem (Gain-Scheduling of OSP subcontrollers)

Consider N OSP subcontrollers $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N$ of the form $\mathbf{y}_i(t) = (\mathcal{G}_i \mathbf{u}_i)(t)$ satisfying

$$\langle \mathbf{u}_i, \mathbf{y}_i \rangle_T \geq \beta_i + \varepsilon_i \|\mathbf{y}_i\|_{2T}^2, \qquad \forall \mathbf{u}_i \in \mathcal{L}_{2e}, \quad \forall T \in \mathbb{R}_{\geq 0},$$

for $i \in \mathcal{N} = \{1, \ldots, N\}$ with $\beta_i \in \mathbb{R}_{\leq 0}$ and $\varepsilon_i \in \mathbb{R}_{>0}$.

*Provided the scheduling matrices are **active**, the gain-scheduled controller $\bar{\mathcal{G}}$ is OSP with*

$$\langle \mathbf{u}_c, \mathbf{y}_c \rangle_T \geq \bar{\beta} + \bar{\varepsilon} \|\mathbf{y}_c\|_{2T}^2, \qquad \forall \mathbf{u}_c \in \mathcal{L}_{2e}, \quad \forall T \in \mathbb{R}_{\geq 0},$$

where

$$\bar{\beta} = \sum_{i \in \mathcal{N}} \alpha_i \beta_i \leq 0, \qquad \bar{\varepsilon} = \frac{\varepsilon_{\min}}{\alpha_{\max}^2 \bar{\sigma}_{\mathbf{\Psi}}^2} > 0, \qquad \varepsilon_{\min} = \min_{i \in \mathcal{N}} \alpha_i \varepsilon_i > 0, \qquad \bar{\sigma}_{\mathbf{\Psi}} = \sup_{t \in [0,T]} \sigma_{\mathbf{\Psi}}(t) > 0,$$

and $\sigma_{\mathbf{\Psi}}(t)$ is the largest singular value of $\mathbf{\Psi}(t) = \begin{bmatrix} \mathbf{\Phi}_1(t)^\mathsf{T} & \ldots & \mathbf{\Phi}_N(t)^\mathsf{T} \end{bmatrix}$.

Discussion

Review (Theorem 1: ISP Case)

*$\bar{\mathcal{G}}$ is ISP if each $\mathcal{G}_i$ is ISP and the scheduling matrices are **strongly active**.*

Review (Theorem 2: OSP Case)

*$\bar{\mathcal{G}}$ is OSP if each $\mathcal{G}_i$ is OSP and the scheduling matrices are **active**.*

Remark (VSP Case)

Consider N VSP subcontrollers $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N$.

- *Each $\mathcal{G}_i$ is also **ISP and OSP simultaneously**.*
- *The condition required for $\bar{\mathcal{G}}$ to be ISP is **more restrictive** than that for OSP.*

*Therefore, the **matrix-gain-scheduling** of N VSP subcontrollers satisfies Theorem 1 and Theorem 2 **simultaneously**, provided the scheduling matrices are **strongly active**.*

Comparison with Existing Literature

(a) Gain-scheduling ISP systems
[Damaren, 1996] and [Forbes, Damaren, 2010]

The scheduling signals, $s_i(t)$, are assumed to satisfy:

1. $\sum_{i \in \mathcal{N}} s_i^2(t) > 0$,
2. $s_i \in \mathcal{L}_{2e} \cap \mathcal{L}_\infty$.

The gain-scheduled controller is ISP with coefficient

$$\hat{\delta} = \inf_{t \in [0,T]} \sum_{i \in \mathcal{N}} s_i^2(t) \delta_{\min},$$

with $\delta_{\min} = \min_{i \in \mathcal{N}} \delta_i$.

(b) Matrix-gain-scheduling ISP systems

The scheduling matrices, $\mathbf{\Phi}_i(t)$ are assumed to satisfy:

1. Strongly active: $\sum_{i \in \mathcal{N}} \nu_i^2(t) > 0$,
2. Bounded: $\sup_{t \in [0,T]} \|\mathbf{\Phi}_i(t)\|_2^2 < \infty$.

The gain-scheduled controller is ISP with coefficient

$$\hat{\delta} = \inf_{t \in [0,T]} \sum_{i \in \mathcal{N}} \nu_i^2(t) \delta_{\min},$$

with $\delta_{\min} = \min_{i \in \mathcal{N}} \alpha_i \delta_i$.

Example: Control of a Two-Link Robotic Manipulator

The control objective is to have the two-link robot track the position and rate trajectory in Figure 2.

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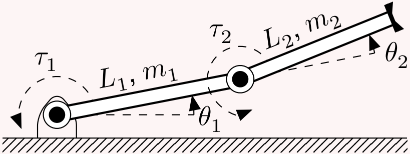


Figure 6: Rigid two-link robotic manipulator.

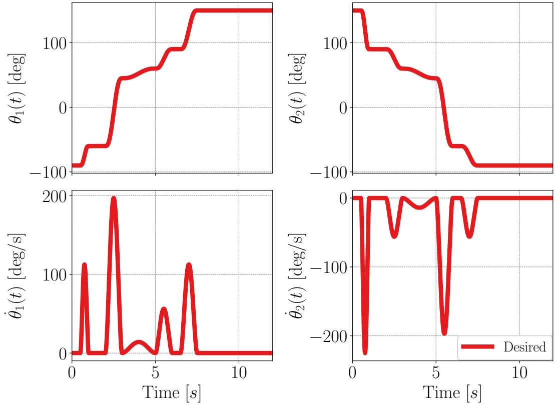


Figure 7: Desired angle position and rate.

Control Synthesis Revisited

- The subcontrollers to be gain scheduled will be **SPR** controllers with **feedthrough**, which are in turn **VSP**.
- The **KYP lemma** and gain matrix $\mathbf{K}$ from the **LQR** problem are used to synthesize the SPR controllers [Benhabib et al., 1981].
- The LQR problem requires the linearized system dynamics.
- Since the SPR controller is a rate-based controller, a **proportional control prewrap** is then added to the system to control the joint displacements of the system.
  - This prewrap does not violate the passive map of the system [Márquez, 2003].

Control Synthesis

- The linearization of the **prewrapped model** about $\bar{\mathbf{q}}_i$ is given by

$$\delta\dot{\mathbf{x}}(t) = \mathbf{A}_i\delta\mathbf{x}(t) + \mathbf{B}_i\delta\mathbf{u}(t), \quad \delta\mathbf{y}(t) = \mathbf{C}_i\delta\mathbf{x}(t),$$

with

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\bar{\mathbf{M}}^{-1}(\bar{\mathbf{q}}_i)\mathbf{K}_\mathrm{p} & \mathbf{0} \end{bmatrix}, \ \mathbf{B}_i = \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{M}}^{-1}(\bar{\mathbf{q}}_i) \end{bmatrix}, \ \mathbf{C}_i = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}^\mathsf{T},$$

where $\bar{\mathbf{M}}(\bar{\mathbf{q}}_i)$ is the **measured mass matrix** and $\mathbf{K}_\mathrm{p}$ is the proportional gain matrix.

- The gain matrix $\mathbf{K}_i$ is computed for each linearization point by solving the algebraic Riccati equation (ARE).

Table 1: Two-Link Manipulator Properties

Link Parameters	Link 1	Link 2
Length [m]	$L_1 = 1.10$	$L_2 = 0.85$
Measured Length [m]	$\bar{L}_1 = 1.08$	$\bar{L}_2 = 0.83$
Mass [kg]	$m_1 = 0.40$	$m_2 = 0.90$
Measured Mass [kg]	$\bar{m}_1 = 0.44$	$\bar{m}_2 = 0.99$

Table 2: Controller Design Parameters

Properties	Symbol	Value
Proportional Gain	$\mathbf{K}_\mathrm{p}$	$\mathrm{diag}(35, 35)$
LQR Weights	$\mathbf{Q}_\mathrm{LQR}$ $\mathbf{R}_\mathrm{LQR}$	$\mathrm{diag}(0.33, 0.25, 180, 180)^{-2}$ $\mathrm{diag}(15, 15)^{-2}$
Feedthrough	δ	0.0001

Control Synthesis

- SPR control synthesis is then completed by using the KYP lemma to set

$$\mathbf{A}_{c,i} = \mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i, \qquad \mathbf{B}_{c,i} = \mathbf{P}_i^{-1}\mathbf{K}_i^{\mathsf{T}}, \qquad \mathbf{C}_{c,i} = \mathbf{K}_i,$$

where $\mathbf{P}_i = \mathbf{P}_i^{\mathsf{T}} \succ 0$ is the solution to the Lyapunov equation, $\mathbf{A}_{c,i}^{\mathsf{T}}\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_{c,i} = -\mathbf{Q}_i$, for $\mathbf{Q}_i = \mathbf{Q}_i^{\mathsf{T}} \succ 0$ [Benhabib et al., 1981].

- To make the SPR controller VSP, an arbitrary small feedthrough term $\mathbf{D}_c = \delta\mathbf{1}$ is added.

- Therefore, for each linearization point $\bar{\mathbf{q}}_i$, a VSP controller, $\mathcal{G}_i : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, can be synthesized with the state-space form

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}_{c,i}\mathbf{x}_i(t) + \mathbf{B}_{c,i}\mathbf{u}_i(t), \qquad \mathbf{y}_i(t) = \mathbf{C}_{c,i}\mathbf{x}_i(t) + \mathbf{D}_c\mathbf{u}_i(t).$$

Scheduling Signals

For the three linearization points $\bar{\mathbf{q}}_1$, $\bar{\mathbf{q}}_2$, and $\bar{\mathbf{q}}_3$, the scheduling signals in Figure 8 are defined as

$$s_1(t) = \begin{cases} 1 - \left(\frac{t}{3}\right)^4 & 0.0 \leq t \leq 3.0, \\ 0 & 3.0 < t, \end{cases} \tag{3a}$$

$$s_2(t) = \begin{cases} 1 - \left(\frac{t-3}{2.8}\right)^4 & 0.2 \leq t \leq 5.8, \\ 0 & \text{otherwise}, \end{cases} \tag{3b}$$

$$s_3(t) = \begin{cases} 0 & 0.0 \leq t < 5.0, \\ 1 - \left(\frac{t-7.5}{2.5}\right)^4 & 5.0 \leq t \leq 7.0, \\ 1 & 7.0 < t. \end{cases} \tag{3c}$$

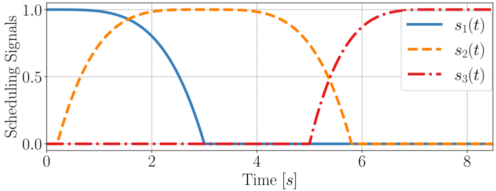


Figure 8: Scalar scheduling signals $s_1(t)$, $s_2(t)$, and $s_3(t)$ defined in (3).

Remark

As required by [Damaren, 1996] and [Forbes, Damaren, 2010], all scheduling signals are bounded, and at all times, at least one scheduling signal is active.

Scheduling Matrices

For $\mathbf{u} : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$, the scheduling of each subcontroller $\mathcal{G}_i$ requires **five hyperparameters**:

- one α_i,
- four scheduling signals for the scheduling matrix $\mathbf{\Phi}_i \in \mathbb{R}^{2\times 2}$.

Using three subcontrollers, one such set of scheduling matrices are

$$\mathbf{\Phi}_1(t) = \begin{bmatrix} 2s_1(t) + 4s_2(t) & 0 \\ 0 & s_1(t) \end{bmatrix}, \qquad \alpha_1 = 2, \tag{4a}$$

$$\mathbf{\Phi}_2(t) = \begin{bmatrix} s_2(t) & 0 \\ s_2(t) & s_2(t) \end{bmatrix}, \qquad \alpha_2 = 1, \tag{4b}$$

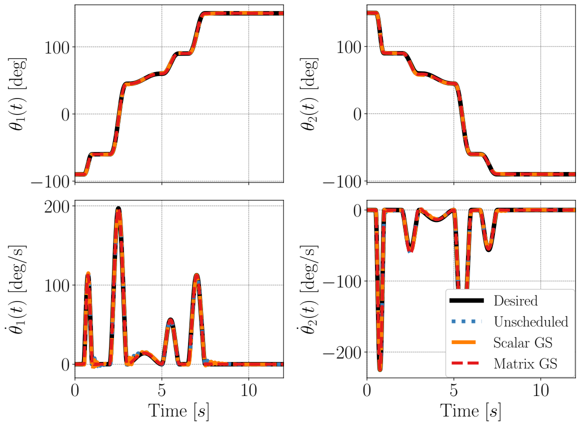
$$\mathbf{\Phi}_3(t) = \begin{bmatrix} s_3(t) + 2s_2(t) & 0 \\ 0 & s_3(t) \end{bmatrix}, \qquad \alpha_3 = 2, \tag{4c}$$

where $s_1(t)$, $s_2(t)$, and $s_3(t)$ are defined in (3).

Results

Three different control approaches:

1. **Unscheduled**: A single VSP controller designed about the linearization of the robot at the end of its trajectory.
2. **Scalar GS**: Three VSP subcontrollers scheduled as per [Forbes, Damaren, 2010]
3. **Matrix GS**: Three VSP subcontrollers scheduled using proposed matrix-gain-scheduling architecture.



Remark

Across all three control approaches, the exact same $\mathbf{K}_\mathrm{p}$, $\mathbf{Q}_\mathrm{LQR}$, $\mathbf{R}_\mathrm{LQR}$, *and* δ *are used for the synthesis of the VSP subcontrollers.*

Results

Table 3: RMS Error of Joint Angle and Joint Angle Rate

Control method	RMS angle error [deg]		RMS angle rate error [deg/s]	
	e_1	e_2	$\dot{e}_1$	$\dot{e}_2$
Unscheduled	0.8328	0.6688	2.5933	1.5587
Scalar scheduling	0.6839	0.6464	2.1307	1.2702
Matrix scheduling	**0.0668**	**0.4515**	**0.1480**	**1.1352**

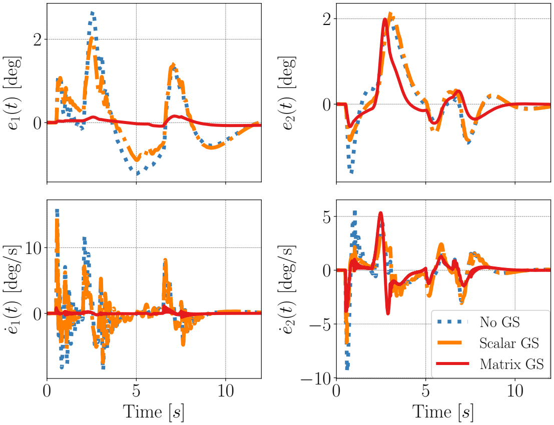
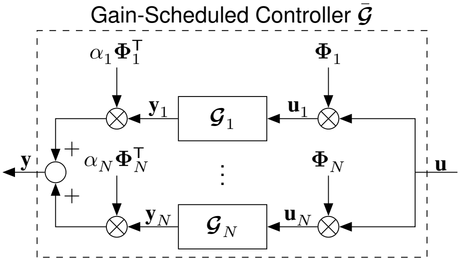


Figure 9: Comparison of joint angles errors and error rates.

Summary



- Proposed a novel gain-scheduling architecture using **scheduling matrices**.
- The gain-scheduled controller is
  - **ISP** if each $\mathcal{G}_i$ is ISP and the scheduling matrices are **strongly active**.
  - **OSP** if each $\mathcal{G}_i$ is OSP and the scheduling matrices are **active**.
  - **VSP** if each $\mathcal{G}_i$ is VSP and the scheduling matrices are **strongly active**.
- The conditions on the scheduling matrices can be interpreted as an extension of the conditions on the scheduling signals reported in [Damaren, 1996] and [Forbes, Damaren, 2010].

Ongoing and Future Work

- The gain-scheduling of a more general class of passive systems is considered in:
  - *Conic Gain-Scheduled Control of an Aeroelastic Airfoil* [Caverly, Brown, 2021],
  - *Gain-Scheduled QSR-Dissipative Systems: An Input-Output Approach* [Anderson, Caverly, Lamperski, 2023].

  These papers also use the same scalar-gain-scheduling architecture presented in [Damaren, 1996].

- In our TAC paper under review, the proposed matrix-gain-scheduling architecture can be extended to consider the gain-scheduling of QSR-dissipative systems for
  - Case 1: All the N subsystems are QSR-dissipative with $\mathbf{Q}_i = \mathbf{Q}_i^\mathsf{T} \prec 0$.
  - Case 2: All the N subsystems are QSR-dissipative with either $\mathbf{Q}_i = \mathbf{Q}_i^\mathsf{T} \prec 0$ or $\mathbf{Q}_i = \mathbf{Q}_i^\mathsf{T} \preceq 0$, and share a common $\mathbf{S}_i = \mathbf{S} \in \mathbb{R}^{n_y \times n_u}$.

- How to optimally design the scheduling matrices?

- Can α_i be time varying?

Questions?

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Backup Slides: Control Effort

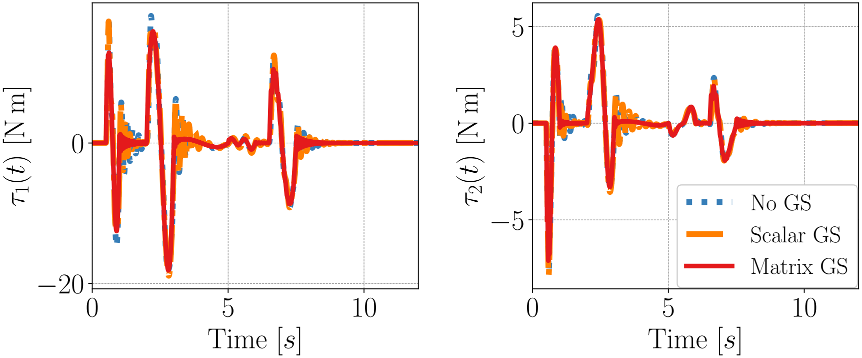

Figure 10: Comparison of joint torques.

Backup Slides: Negative Feedback Interconnection

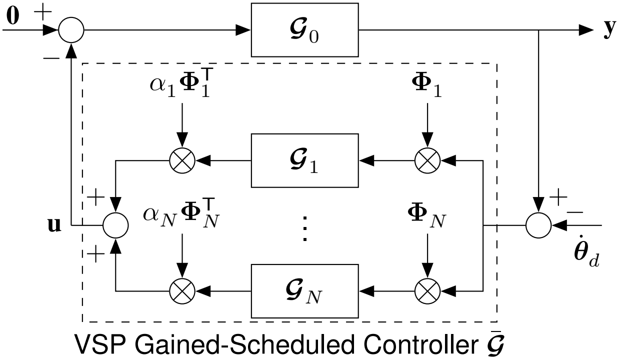

Figure 11: Gain-scheduled feedback control of the plant to be controlled $\mathcal{G}_0$, prewrapped with proportional control, and the gain-scheduled controller $\bar{\mathcal{G}}$.

Backup Slides: Trajectory Generation

Trajectory generation is achieved similar to [Forbes, Damaren, 2010], by choosing discrete joint angles $\boldsymbol{\theta}_{\mathrm{d}}(t_k)$ and $\boldsymbol{\theta}_{\mathrm{d}}(t_{k+1})$ at times t_k and t_{k+1}, and interpolating between them as such

$$\eta(t) = \frac{t - t_k}{t_{k+1} - t_k}, \tag{5a}$$

$$p_5(t) = 6\eta^5 - 15\eta^4 + 10\eta^3, \tag{5b}$$

$$\boldsymbol{\theta}_{\mathrm{d}}(t) = p_5(t)\big(\boldsymbol{\theta}_{\mathrm{d}}(t_{k+1}) - \boldsymbol{\theta}_{\mathrm{d}}(t_k)\big) + \boldsymbol{\theta}_{\mathrm{d}}(t_k). \tag{5c}$$

The desired discrete joint angles are chosen such that the joint angles operate within $[-90°, 150°]$.

Backup Slides: Expanding OSP Coefficient for base case

The special case of $\alpha_i = 1$ and $\mathbf{\Phi}_i(t) = s_i(t)\mathbf{1}$ is referred to as the *base case*.
As discussed in our TAC paper under review, for the base case, the gain-scheduled OSP coefficient $\bar{\varepsilon}$ can be expanded as

$$\bar{\varepsilon} = \frac{\varepsilon_{\min}}{\alpha_{\max}^2 \bar{\sigma}_{\mathbf{\Psi}}^2} = \frac{\varepsilon_{\min}}{\sup_{t\in[0,T]} \sum_{i\in\mathcal{N}} |s_i(t)|^2} > 0. \tag{6}$$

Backup Slides: Finite $\mathcal{L}_2$ Gain

Using the Cauchy–Schwartz inequality, it can be shown that if a system is OSP with $\mathbf{Q} = -\varepsilon\mathbf{1}$, it also possesses finite $\mathcal{L}_2$ gain such that $\varepsilon = 1/\gamma$. Here, by defining $\varepsilon_i = 1/\gamma_i$ with $\gamma_i \in \mathbb{R}_{>0}$, (6) can be rewritten as $\varepsilon = 1/\gamma$, where

$$\gamma = \max_{i \in \mathcal{N}}(\gamma_i) \sup_{t \in [0,T]} \sum_{i \in \mathcal{N}} |s_i(t)|^2 > 0. \tag{7}$$

Therefore, the gain-scheduled system $\bar{\mathcal{G}}$ posses finite $\mathcal{L}_2$ gain with γ defined as per (7).
In [Forbes, Damaren, 2010], it is shown that the gain-scheduled controller has finite $\mathcal{L}_2$ gain given by

$$\gamma = \sum_{i \in \mathcal{N}} \|s_i\|_\infty^2 \gamma_i > 0. \tag{8}$$